

NONUNITARY AND UNITARY APPROACH TO EIGENVALUE PROBLEM OF BOSON OPERATORS AND SQUEEZED COHERENT STATES

Wünsche, A.
Arbeitsgruppe "Nichtklassische Strahlung"
der Max-Planck-Gesellschaft an der
Humboldt-Universität Berlin
Rudower Chaussee 5, O-1199 Berlin, Germany

Abstract

The eigenvalue problem of the operator $a + \zeta a^\dagger$ is solved for arbitrary complex ζ by applying a nonunitary operator to the vacuum state. This nonunitary approach is compared with the unitary approach leading for $|\zeta| < 1$ to squeezed coherent states.

1 Introduction

The eigenvalue problem to linear combinations of boson operators in the standardized form $a + \zeta a^\dagger$ can be solved with squeezed coherent states only in the case $|\zeta| < 1$ when it is equivalent to the eigenvalue problem of an operator $\kappa a + \mu a^\dagger$ under the condition $|\kappa|^2 - |\mu|^2 = 1$ with the substitution $\zeta \equiv \frac{\mu}{\kappa}$, (ζ, κ, μ arbitrary complex numbers) [1]. This corresponds to the unitary approach because the squeezed coherent states can be obtained by applying unitary squeezing operators to coherent states [2]. However, this eigenvalue problem can also be solved for arbitrary complex ζ with a nonunitary approach providing in the limiting case $\zeta \rightarrow \infty$ even the solution of the eigenvalue problem for the boson creation operator a^\dagger . The corresponding eigenstates are not normalizable for $|\zeta| \geq 1$ and are not states of the usual Hilbert space H (Fock space) in this case but they are states of a rigged Hilbert space K' in Gelfand triplets of spaces $K \subset H \subset K'$ [3]. Such states that do not give finite expectation values for relevant operators as, for example, for the number operator could be, therefore, considered as pathological ones. However, they play an important auxiliary role for the formulation of a new kind of orthogonality and completeness relations on paths through the complex plane of eigenvalues, where at once two dual states belonging to the parameters ζ and $\zeta' = \frac{1}{\zeta^*}$ or $\zeta\zeta'^* = 1$ are involved [1].

2 Nonunitary approach to the eigenvalue problem

The solution of the eigenvalue problem

$$(a + \zeta a^\dagger)|\alpha; \zeta\rangle = \alpha|\alpha; \zeta\rangle, \quad (1)$$

can be represented in the number-state basis $|n\rangle$ in the following nonnormalized form

$$\begin{aligned} |\alpha; \zeta\rangle &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} H_n \left(\frac{\alpha}{\sqrt{2\zeta}} \right) \left(\frac{\sqrt{2\zeta}}{2} \right)^n |n\rangle \\ &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} \left(\frac{\zeta}{2\alpha^2} \right)^k |n\rangle, \end{aligned} \quad (2)$$

or by derivatives of a Gaussian function in the form

$$|\alpha; \zeta\rangle = \exp \left(\frac{\alpha^2}{2\zeta} \right) \sum_{n=0}^{\infty} \frac{(-\zeta)^n}{\sqrt{n!}} \frac{\partial^n}{\partial \alpha^n} \exp \left(-\frac{\alpha^2}{2\zeta} \right) |n\rangle. \quad (3)$$

Substituting in (2) the number states by the generation from the vacuum state, one obtains by means of the generating function of the Hermite polynomials $H_n(z)$

$$|\alpha; \zeta\rangle = \exp \left(\alpha a^\dagger - \frac{\zeta}{2} a^{\dagger 2} \right) |0\rangle. \quad (4)$$

Two special cases are easily obtained from these formula, the coherent states $|\alpha; 0\rangle$

$$|\alpha; 0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \equiv \exp \left(\frac{\alpha \alpha^*}{2} \right) |\alpha\rangle, \quad (5)$$

and the squeezed vacuum states $|0; \zeta\rangle$

$$|0; \zeta\rangle = \sum_{m=0}^{\infty} \sqrt{\frac{(2m-1)!!}{2^m m!}} (-\zeta)^m |2m\rangle. \quad (6)$$

The nonunitary operator $\exp \left(\alpha a^\dagger - \frac{\zeta}{2} a^{\dagger 2} \right)$ does not preserve the normalization of the states. The corresponding normalized states

$$|\alpha; \zeta\rangle_{\text{norm}} = (1 - \zeta \zeta^*)^{-\frac{1}{2}} \exp \left\{ -\frac{2\alpha \alpha^* - (\zeta^* \alpha^2 + \zeta \alpha^{*2})}{4(1 - \zeta \zeta^*)} \right\} |\alpha; \zeta\rangle, \quad (7)$$

are only possible for $|\zeta| < 1$.

The expectation values of the canonical operators

$$Q(\varphi) \equiv \sqrt{\frac{\hbar}{2}} (a e^{i\varphi} + a^\dagger e^{-i\varphi}), \quad P(\varphi) \equiv -i \sqrt{\frac{\hbar}{2}} (a e^{i\varphi} - a^\dagger e^{-i\varphi}), \quad (8)$$

denoted by cross-lines are

$$\overline{Q(\varphi)} = \sqrt{\frac{\hbar}{2}} \frac{\alpha e^{i\varphi} (1 - \zeta^* e^{-i2\varphi}) + \alpha^* e^{-i\varphi} (1 - \zeta e^{i2\varphi})}{1 - \zeta \zeta^*},$$

$$\overline{P(\varphi)} = -i\sqrt{\frac{\hbar}{2}} \frac{\alpha e^{i\varphi}(1 + \zeta^* e^{-i2\varphi}) - \alpha^* e^{-i\varphi}(1 + \zeta e^{i2\varphi})}{1 - \zeta\zeta^*}, \quad (9)$$

and their variances are

$$\begin{aligned} \overline{(\Delta Q(\varphi))^2} &= \frac{\hbar}{2} \frac{(1 - \zeta e^{i2\varphi})(1 - \zeta^* e^{-i2\varphi})}{1 - \zeta\zeta^*}, \\ \overline{(\Delta P(\varphi))^2} &= \frac{\hbar}{2} \frac{(1 + \zeta e^{i2\varphi})(1 + \zeta^* e^{-i2\varphi})}{1 - \zeta\zeta^*}, \end{aligned} \quad (10)$$

The uncertainty product

$$\overline{(\Delta Q(\varphi))^2} \overline{(\Delta P(\varphi))^2} = \frac{\hbar^2}{4} \left\{ 1 - \frac{(\zeta e^{i2\varphi} - \zeta^* e^{-i2\varphi})^2}{(1 - \zeta\zeta^*)^2} \right\}, \quad (11)$$

is the minimal possible one for 4 angles φ_{ext} according to

$$e^{i4\varphi_{ext}} = \frac{\zeta^*}{\zeta}, \quad \overline{(\Delta Q(\varphi_{ext}))^2} \overline{(\Delta P(\varphi_{ext}))^2} = \frac{\hbar^2}{4}, \quad (12)$$

corresponding to the extreme values of the variances

$$\overline{(\Delta Q(\varphi_{ext}))^2} = \frac{\hbar}{2} \frac{1 \mp |\zeta|}{1 \pm |\zeta|}, \quad \overline{(\Delta P(\varphi_{ext}))^2} = \frac{\hbar}{2} \frac{1 \pm |\zeta|}{1 \mp |\zeta|}. \quad (13)$$

One has pure amplitude (phase) squeezing if the minimal (maximal) value of $\overline{(\Delta Q(\varphi_{ext}))^2}$ corresponds to $\overline{P(\varphi_{ext})} = 0$. This leads to the following coordinate-invariant conditions for the arguments of the Hermite polynomials in (2):

1. amplitude squeezing, $\frac{\alpha}{\sqrt{2\zeta}}$ real numbers,
2. phase squeezing, $\frac{\alpha}{\sqrt{2\zeta}}$ imaginary numbers.

In the more general case, $\frac{\alpha}{\sqrt{2\zeta}}$ is a complex number. The expectation value of the number operator is

$$\overline{N} = \left(\frac{\alpha - \zeta\alpha^*}{1 - \zeta\zeta^*} \right) \left(\frac{\alpha^* - \zeta^*\alpha}{1 - \zeta\zeta^*} \right) + \frac{\zeta\zeta^*}{1 - \zeta\zeta^*}. \quad (14)$$

and its variance

$$\overline{(\Delta N)^2} = \frac{((1 + \zeta\zeta^*)\alpha - 2\zeta\alpha^*)((1 + \zeta\zeta^*)\alpha^* - 2\zeta^*\alpha)}{(1 - \zeta\zeta^*)^3} + \frac{2\zeta\zeta^*}{(1 - \zeta\zeta^*)^2}. \quad (15)$$

The nonunitary approach provides a new convenient parametrization of the squeezed coherent states.

3 Unitary approach to the eigenvalue problem

The unitary squeezing operators

$$S(\xi, \eta = \eta^*, \xi^*) = \exp \left\{ \xi \frac{1}{2} a^2 - \xi^* \frac{1}{2} a^{\dagger 2} + i\eta \frac{1}{2} (aa^\dagger + a^\dagger a) \right\} \quad (16)$$

transform the basis operators a and a^\dagger according to

$$S(\xi, \eta, \xi^*)(a, a^\dagger)(S(\xi, \eta, \xi^*))^\dagger = (a, a^\dagger) \begin{pmatrix} \kappa, \mu^* \\ \mu, \kappa^* \end{pmatrix}$$

$$\kappa = ch\epsilon - i\eta \frac{sh\epsilon}{\epsilon}, \mu = \xi^* \frac{sh\epsilon}{\epsilon}, \epsilon \equiv \sqrt{|\xi|^2 - \eta^2}, |\kappa|^2 - |\mu|^2 = 1. \quad (17)$$

The solution of the eigenvalue problem of the operator $a + \zeta a^\dagger$ is obtained by the following application of the unitary squeezing operators to coherent states $|\gamma\rangle$

$$\begin{aligned} \frac{e^{i\chi}}{\sqrt{1-|\zeta|^2}} (a + \zeta a^\dagger) S(\xi, \eta, \xi^*) \left| \frac{e^{i\chi}\alpha}{\sqrt{1-|\zeta|^2}} \right\rangle &= (\kappa a + \mu a^\dagger) S(\xi, \eta, \xi^*) |\gamma\rangle \\ &= \frac{e^{i\chi}\alpha}{\sqrt{1-|\zeta|^2}} S(\xi, \eta, \xi^*) \left| \frac{e^{i\chi}\alpha}{\sqrt{1-|\zeta|^2}} \right\rangle \end{aligned} \quad (18)$$

where χ is an arbitrary angle and ξ and η are given by

$$\begin{aligned} \xi &= \frac{e^{-i\chi}\zeta^*}{\sqrt{1-|\zeta|^2}} \theta, \eta = -\frac{\sin \chi}{\sqrt{1-|\zeta|^2}} \theta, \\ \theta &\equiv \sqrt{\frac{1-|\zeta|^2}{|\zeta|^2 - \sin^2 \chi}} \text{Arsh} \sqrt{\frac{|\zeta|^2 - \sin^2 \chi}{1-|\zeta|^2}}. \end{aligned} \quad (19)$$

By choosing $\chi = 0$ one finds

$$\begin{aligned} |\alpha; \zeta\rangle_{\text{norm}} &= \exp \left\{ -\frac{\zeta^* \alpha^2 - \zeta \alpha^{\ast 2}}{4(1-|\zeta|^2)} \right\} \\ &\cdot \exp \left\{ \frac{1}{2|\zeta|} \text{Arsh} \left(\frac{|\zeta|}{\sqrt{1-|\zeta|^2}} \right) (\zeta^* a^2 - \zeta a^{\dagger 2}) \right\} \left| \frac{\alpha}{\sqrt{1-|\zeta|^2}} \right\rangle. \end{aligned} \quad (20)$$

The unitary approach is restricted to $|\zeta| < 1$.

4 Dual states and eigenstates of the creation operator

The states $\langle \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*} |$ are left eigenstates to the operator $a + \zeta a^\dagger$ according to

$$\langle \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*} | (a + \zeta a^\dagger) = \beta \langle \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*} |, \quad (21)$$

and they are dual to the states $|\alpha; \zeta\rangle$ in the sense of the orthogonality relation

$$\langle \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*} | \alpha; \zeta \rangle = \sqrt{2\pi\zeta} \exp\left(\frac{\alpha^2}{2\zeta}\right) \delta(\alpha - \beta), \quad (22)$$

and of the completeness relation

$$\frac{1}{\sqrt{2\pi\zeta}} \int_C da \exp\left(-\frac{\alpha^2}{2\zeta}\right) |\alpha; \zeta\rangle \langle \frac{\alpha^*}{\zeta^*}; \frac{1}{\zeta^*}| = I. \quad (23)$$

The integration path C through the complex plane is widely arbitrary with the only restriction that it must begin in one sector and end in the other sector where $\exp\left(-\frac{\alpha^2}{2\zeta}\right)$ vanishes in the infinity for fixed values of ζ .

The eigenstates of the creation operator a^\dagger according to

$$a^\dagger |\tilde{\beta}; \infty\rangle = \beta |\tilde{\beta}; \infty\rangle \quad (24)$$

are

$$|\tilde{\beta}; \infty\rangle = \exp\left(-a^\dagger \frac{\partial}{\partial \beta}\right) \delta(\beta) |0\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \frac{\partial^n}{\partial \beta^n} \delta(\beta) |n\rangle \quad (25)$$

where $\delta(\beta)$ is the one-dimensional delta function of complex argument (analytic functional). They are orthogonal to the coherent states $|\alpha; 0\rangle$

$$\langle \alpha^*; 0 | \tilde{\beta}; \infty \rangle = \delta(\alpha - \beta). \quad (26)$$

This relation shows also that the coherent states are already complete on paths through the complex plane.

More details and references can be found in [1].

References

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